International Journal of Theoretical Physics, Vol. 30, No. 4, 1991

Uncertainties and an Interpretation of Nonrelativistic Quantum Theory

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Received July 9, 1990

We propose an interpretation of nonrelativistic quantum theory which can be considered a generalized Copenhagen interpretation. The uncertainties (i.e., Δq and Δp) in Heisenberg's uncertainty relation $\Delta q \cdot \Delta p \geq \hbar/2$ can be characterized as (average) errors in an approximate simultaneous measurement if the interpretation proposed here is accepted in nonrelativistic quantum mechanics. Under this interpretation, the (discrete) trajectory of a particle (like "Wilson chamber") is significant enough. We propose to analyze this trajectory numerically.

1. INTRODUCTION

Recently, we discussed Heisenberg's uncertainty relation $\Delta q \cdot \Delta p \geq \hbar/2$ (Ishikawa, 1991), gave a mathematical definition of Δq and Δp, and proved a certain inequality which could be considered as a mathematical representation of Heisenberg's uncertainty relation. We mention some of the results obtained in Ishikawa (1991) in order to exhibit our motivations in this note.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$. Let $A_0, A_1, \ldots, A_{N-1}$ be any physical quantities (i.e., self-adjoint operators) in a Hilbert space H. A quarter $\mathbf{M} = (K, v, (X, \mathcal{F}, F), f = (f_0, \ldots, f_{N-1}))$ is called an approximate simultaneous measurements of $\{A_k\}_{k=0}^{N-1}$ in H if it satisfies the following conditions:

1. v is an element in a Hilbert space K such that $\|v\|_K = 1$, and $(X, \mathcal{F}, F)$ is a projection-valued probability space in a tensor Hilbert space $H \otimes K$ and $f: X \to \mathbf{R}^N$ is a measurable map.

2. Put $\hat{A}_k = \int_X f_k(x) F(dx) (k = 0, 1, \ldots, N-1)$; then, for each k, a set $D_v(\hat{A}_k)$ $(\equiv \{u \in H: u \otimes v \in D(\hat{A}_k)$, the domain of $\hat{A}_k\})$ is a core of A_k, i.e., A_k is essentially self-adjoint on $D_v(\hat{A}_k)$.

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0020-7748/91/0400-0401$06.50/0

3. For each k, $\langle u, A_k u\rangle_H = \langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K}$ $(u \in D_v(\hat{A}_k))$.

Furthermore, we assume that **M** satisfies the following additional conditions:

4. For each k, $(\hat{A}_k - A_k\otimes I)$ on $D(\hat{A}_k)\cap D(A_k\otimes I)$ has the unique self-adjoint extension $[\hat{A}_k - A_k\otimes I]$.

5. A set $\{u\in H: u\otimes v \in \bigcap_{k=0}^{N-1} D(\hat{A}_k)\}$ is a dense set in a Banach space H_0 $(\equiv \{u\in H: u\otimes v\in \bigcap_{k=0}^{N-1} D([\hat{A}_k - A_k\otimes I])\})$ with the norm

$$\|u\|_{H_0}\left(\equiv \|u\|_H + \sum_{k=0}^{N-1} \|[\hat{A}_k - A_k\otimes I](u\otimes v)\|_{H\otimes K}\right)$$

Note that the existence of an approximate simultaneous measurement of $\{A_k\}_{k=0}^{N-1}$ satisfying conditions 4 and 5 is proved in Abu-Zeid (1987) [or Ishikawa (1991) in detail]).

Now the unfitness [in Ishikawa (1991) we did not dare to call it "error" or "uncertainty" since its physical meaning seemed to be not clear] $\{\Delta_{\mathbf{M}}(A_k, u)\colon k=0,\ldots,N-1\}$ of an approximate simultaneous measurement **M** for $\{A_k\}_{k=0}^{N-1}$ on a state u ($\|u\|_H=1$) is defined by $\Delta_{\mathbf{M}}(A_k, u)\equiv \|[\hat{A}_k - A_k\otimes I](u\otimes v)\|_{H\otimes K}$ if $u\otimes v\in D([\hat{A}_k - A_k\otimes I]) = \infty$ otherwise. We obtained the following theorem in Ishikawa (1991).

Theorem 1. Let A_0 and A_1 be a pair of conjugate observables in a Hilbert space H (i.e., symbolically, $A_0A_1 - A_1A_0 = i\hbar$). Let $\mathbf{M} = (K, v, (X, \mathcal{F}, F), f=(f_0, f_1))$ be any approximate simultaneous measurement of A_0 and A_1 satisfying the additional conditions 4 and 5. Then, the following inequality holds:

$$\Delta_{\mathbf{M}}(A_0, u)\cdot \Delta_{\mathbf{M}}(A_1, u) \geq \hbar/2 \tag{1}$$

for all $u\in H(\|u\|_H=1)$, where the left-hand side of (1) is defined as $=\infty$ if $\Delta_{\mathbf{M}}(A_k, u)=\infty$ for some $k=0, 1$.

Special and simple cases of this theorem were also investigated in Ali and Emch (1974), Ali and Prugovečki (1976), and Busch (1985).

If we take a standpoint within the so-called "Copenhagen interpretation," we believe that inequality (1) is just Heisenberg's uncertainty relation (though there is a possibility to improve the conditions 4 and 5. And we believe that the relation between the EPR argument (Einstein *et al.*, 1935; Selleri, 1990) and Heisenberg's uncertainty relation becomes clear in Ishikawa (1991). However, the observations in Ishikawa (1991) are not necessarily physical, but rather mathematical. Hence we think that the physical meaning of the unfitness $\{\Delta_{\mathbf{M}}(A_k, u)\colon k=0,1\}$ is still not clear. Furthermore, we think that it is necessary to discuss the matter from various viewpoints in order to conclude that the inequality (1) is a mathematical representation of Heisenberg's uncertainty relation.

It is natural to consider that the "error Δ" in the (approximate) measurement should mean the (average) distance between the "true" value $\hat{x}$ and the (approximate) measurement value x, i.e., $\Delta = d(x, \hat{x})$. Also, the "true" value usually means the value obtained by the (accurate) measurement in the Copenhagen interpretation. However, if we think in this way, the uncertainties (i.e., Δq and Δp) cannot be characterized as "errors." Because the Heisenberg uncertainty relation assures that any accurate simultaneous measurement of the position and momentum does not exist, it is impossible to know the "true" value $\hat{x}$; then we cannot define the "errors" $\Delta(q)$ and $\Delta(p)$. So, in Ishikawa (1991) we did not call $\Delta_{\mathbf{M}}(A_k, u)$ "error," but "unfitness." We think that it is suitable to call it "unfitness" if we study it under the so-called Copenhagen interpretation. However, we think that the physical meaning of the "unfitness" $\{\Delta_{\mathbf{M}}(A_k, u) \mid k = 0, 1\}$ will never become clear within this interpretation. Therefore, in Section 2, we propose an interpretation in nonrelativistic quantum mechanics which can be considered as a generalized Copenhagen interpretation. Under this interpretation, we define the "error" $\{\bar{\delta}_{\mathbf{M}}(A_k, u) \mid k = 0, 1\}$ of an approximate simultaneous measurement $\mathbf{M} = (K, v, (X, \mathcal{F}, F), f = (f_0, f_1))$ of A_0 and A_1 and show that $\{\bar{\delta}_{\mathbf{M}}(A_k, u) \mid k = 0, 1\}$ and $\{\Delta_{\mathbf{M}}(A_k, u) \mid k = 0, 1\}$ are equal under some conditions. And we show that an analogue of Theorem 1 [i.e., $\bar{\delta}_{\mathbf{M}}(A_0, u)\bar{\delta}_{\mathbf{M}}(A_2, u) \geq \hbar/2$] holds. Also, our interpretation seems to be very convenient for discussing quantum mechanical theory. As an example, in Section 3 we analyze the so-called "Wilson chamber" under this interpretation. We show that there is some reason to consider the (discrete) trajectory of a particle. Furthermore, we propose to analyze this trajectory numerically.

2. AN INTERPRETATION AND UNCERTAINTIES

In this section, we first propose an interpretation in nonrelativistic quantum mechanics. Let V be a Hilbert space. A projection-valued probability F on a measurable space X (with a σ-field $\mathcal{F}$) in a Hilbert space V is defined by the following conditions:

6. For every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a projection in V such that $F(\emptyset) = 0$ and $F(X) = I$, where 0 is a 0-operator and I is an identity operator in V.
7. For any countable decomposition $\{\Xi_j\}_{j=1}^{\infty}$ of Ξ, $(\Xi_j, \Xi \in \mathcal{F})$, $F(\Xi) = \sum_{j=1}^{\infty} F(\Xi_j)$ holds, where the series is weakly convergent.
8. $\mathcal{F}$ can be generated by a countable family $\{\Xi_i^0 \mid \Xi_i^0 \subset X, i = 1, 2, \ldots\}$.

Also, a triplet $(X, \mathcal{F}, F)$ is called a projection-valued probability space in V. Note that the condition (8) is rather weak. For example, if X is a complete separable metric space and $\mathcal{F}$ is its Borel space, then (8) is clearly assured.

In this note, we use the projection-valued probability spaces as a mathematical model of observables. So, we shall chiefly call a projection-valued probability space an observable. Note that any self-adjoint operator A in V has the unique spectral representation $A = \int_{\mathbf{R}} \lambda E_A(d\lambda)$; then we sometimes identify a self-adjoint operator A with an observable $(\mathbf{R}, \mathscr{B}, E_A)$, where $\mathscr{B}$ is a Borel field on $\mathbf{R}$, so we sometimes consider $A = (\mathbf{R}, \mathscr{B}, E_A)$. Projection-valued probability spaces (or more generally, positive operator-valued probability spaces) are investigated in Davies (1976), where a positive operator-valued probability space is called an observable. However, we shall here call a projection-valued probability space an observable, since we do not use positive operator-valued probability spaces in this note.

Axiom 0 (Born's probabilistic interpretation). Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathscr{F}, F)$ be an observable in V. Then, the probability that x_0 $(\in X)$, the measurement value obtained by the measurement of the observable $(X, \mathscr{F}, F)$ for this system S, belongs to a set Ξ $(\in \mathscr{F})$ is given by $\langle \psi, F(\Xi)\psi \rangle_V$.

Of course, this axiom is fundamental. The following axiom requires that measurement value$\Rightarrow$"true" value.

Axiom 1 (Measurement value and "true" value). Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathscr{F}, F)$ be an observable in V. If we get x_0 $(\in X)$ by the measurement of the observable $(X, \mathscr{F}, F)$ for this system S, then we can believe that the "true" value of the observable $(X, \mathscr{F}, F)$ for this system S is the same x_0.

All physicists will agree to this axiom. However, our interpretation does not assert that measurement value$\Leftrightarrow$"true" value. Of course, if we think so, we must define "true" value. This will be done implicitly through Postulates 1–5 and Axioms 1 and 2. In this sense, Postulates 1–5 can be regarded as the definition of "true" value. Before we mention our main Axiom 2, we must prepare Postulates 1–4.

Postulate 1. Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be observables in V such that there is a measurable map $f: X \to Y$ satisfying that $F(f^{-1}(\Gamma)) = G(\Gamma)$ $(\forall \Gamma \in \mathscr{G})$. If we know that the "true" value of the observable $(X, \mathscr{F}, F)$ for a system S is x_0 $(\in X)$, then we can believe that the "true" value y_0 $(\in Y)$ of the observable $(Y, \mathscr{G}, G)$ for this system S is $f(x_0)$, that is, $y_0 = f(x_0)$.

Remark 1. Using this postulate, we can define the simultaneous measurement [in the ordinary sense (von Neumann, 1932)] of commutative observables. For example, see the arguments below (5) and (6). Though these arguments are mentioned in the general case, the reader can easily find that observations about simultaneous measurements cannot be made without Postulate 1. So, we believe that almost physicists will agree to Postulate 1, if they think that simultaneous measurements (in the ordinary sense) of commutative observables should be possible.

Before we mention Postulate 2, we must prepare the following lemma.

Lemma 1. Let ψ be a state in a Hilbert space V (i.e., $\psi \in V$ and $\|\psi\|_V = 1$). Let $(X, \mathscr{F}, F)$ be an observable in V. Put $P_{(F,\psi)} \equiv$ "the projection on a smallest closed subspace that contains $\{F(\Xi)\psi \mid \Xi \in \mathscr{F}\}$". Then, the following hold:

$$P_{(F,\psi)}\psi = \psi \tag{2}$$

$$P_{(F,\psi)}F(\Xi) = F(\Xi)P_{(F,\psi)} \qquad (\forall \Xi \in \mathscr{F}) \tag{3}$$

Proof. Clearly, (2) holds. Let $\phi \in V$. Then, for any positive integer n, we can take an n decomposition $\{\Xi_j^n\}_{j=1}^n$ of X ($\Xi_j^n \in \mathscr{F}$) and a complex sequence $\{\alpha_j^n\}_{j=1}^n$ such that $P_{(F,\psi)}\phi = \lim_{n\to\infty} \sum_{j=1}^n \alpha_j^n F(\Xi_j^n)\psi$. Therefore, we see that

$$F(\Xi)P_{(F,\psi)}\phi = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j^n F(\Xi \cap \Xi_j^n)\psi \in P_{(F,\psi)}V$$

Then, $P_{(F,\psi)}F(\Xi)P_{(F,\psi)} = F(\Xi)P_{(F,\psi)}$ holds. This implies that

$$\begin{aligned} P_{(F,\psi)}F(\Xi) &= (F(\Xi)P_{(F,\psi)})^* = (P_{(F,\psi)}F(\Xi)P_{(F,\psi)})^* \\ &= P_{(F,\psi)}F(\Xi)P_{(F,\psi)} = F(\Xi)P_{(F,\psi)} \end{aligned}$$

Therefore, the proof is completed. ■

Postulate 2. Let ψ be a state of a system S in a Hilbert space V. Let $(X, \mathscr{F}, F)$ and $(X, \mathscr{F}, F')$ be observables in V such that

$$P_{(F,\psi)}F(\Xi) = P_{(F',\psi)}F'(\Xi) \qquad (\forall \Xi \in \mathscr{F}) \tag{4}$$

where $P_{(F,\psi)}$ and $P_{(F',\psi)}$ are defined in Lemma 1. If we know that the "true" value of the observable $(X, \mathscr{F}, F)$ for a system S is x_0 ($\in X$), then we can believe that the "true" value x_1 ($\in X$) of the observable $(X, \mathscr{F}, F')$ for this system S is the same x_0, that is, $x_1 = x_0$.

Remark 2. Without this postulate, our arguments in this note are essentially possible (see Remark 5). However, we believe that this postulate must be assumed in quantum mechanics if we cannot find any experiment that is incompatible with Postulate 2.

Next we mention Postulate 3, which is very close to our main Postulate 4. For this, we must study simultaneous measurements.

Write $F \approx_{\psi} F'$ if $P_{(F,\psi)}F(\Xi) = P_{(F',\psi)}F'(\Xi)$ holds for all $\Xi \in \mathscr{F}$. Then, the relation $\approx_{\psi}$ is clearly an equivalent relation. Also, note that $F \approx_{\psi} F'$ implies that $P_{(F,\psi)} = P_{(F',\psi)}$. Let ψ be a state of a system S in a Hilbert space V. Now we shall consider the simultaneous measurement

$$\mathbf{M} = ((Z, \mathscr{M}, M), f_1\colon Z \to X_1, f_2\colon Z \to X_2)$$

of observables $(X_1, \mathscr{F}_1, F_1)$ and $(X_2, \mathscr{F}_2, F_2)$ for a system S with a state ψ (briefly, with respect to ψ), that is, $\mathbf{M}$ has the property that there exist projection-valued probabilities M_1 and M_2 on Z (with a σ-field $\mathscr{M}$) such that

$$M \approx_{\psi} M_1 \approx_{\psi} M_2 \tag{5}$$

for each $i = 1, 2$,

$$f_i\colon Z \to X_i \text{ is a measurable function such that}$$
$$M_i(f_i^{-1}(\Xi_i)) = F_i(\Xi_i) \text{ for all } \Xi_i \in \mathscr{F}_i \tag{6}$$

The reason that $\mathbf{M}$ is called a simultaneous measurement is as follows: if we know the "true" value z of the observable $(Z, \mathscr{M}, M)$ for this system S, then we can believe, by Postulate 2, that the "true" value of observables $(Z, \mathscr{M}, M_1)$ and $(Z, \mathscr{M}, M_2)$ for this system S is the same z. Therefore, we can know, by Postulate 1, that the "true" values of observables $(X_1, \mathscr{F}_1, F_1)$ and $(X_2, \mathscr{F}_2, F_2)$ for this system S are $f_1(z)$ and $f_2(z)$ respectively.

Assume the existence of a simultaneous measurement

$$\mathbf{M} = ((Z, \mathscr{M}, M), f_1\colon Z \to X_1, f_2\colon Z \to X_2)$$

of observables $(X_1, \mathscr{F}_1, F_1)$ and $(X_2, \mathscr{F}_2, F_2)$ for the system S. Then, since $P_{(M,\psi)} = P_{(M_1,\psi)} = P_{(M_2,\psi)}$, we see, by Lemma 1, that

$$\begin{aligned} F_1(\Xi_1)F_2(\Xi_2)P_{(M,\psi)} &= M_1(f_1^{-1}(\Xi_1))M_2(f_2^{-1}(\Xi_2))P_{(M,\psi)} \\ &= M(f_1^{-1}(\Xi_1) \cap f_2^{-1}(\Xi_2))P_{(M,\psi)} \\ &= F_2(\Xi_2)F_1(\Xi_1)P_{(M,\psi)} \qquad (\forall \Xi_1 \in \mathscr{F}_1, \forall \Xi_2 \in \mathscr{F}_2) \end{aligned} \tag{7}$$

so we also see, by (2), that

$$F_1(\Xi_1)F_2(\Xi_2)\psi = F_2(\Xi_2)F_1(\Xi_1)\psi \qquad (\forall \Xi_1 \in \mathscr{F}_1, \forall \Xi_2 \in \mathscr{F}_2) \tag{8}$$

Namely, observables $(X_1, \mathscr{F}_1, F_1)$ and $(X_2, \mathscr{F}_2, F_2)$ commute with respect to ψ. On the contrary, from (8), we can construct a typical simultaneous measurement $((Z, \mathscr{M}, M), f\colon Z \to X_1 \times X_2)$ of observables $(X_1, \mathscr{F}_1, F_1)$ and $(X_2, \mathscr{F}_2, F_2)$ with respect to a state ψ as follows. Put $(X'_i, \mathscr{F}'_i) = (X_i, \mathscr{F}_i)$ $(i = 1, 2)$. Define $Z \equiv (X_1 \times X_2) \cup X'_1 \cup X'_2$ and $\mathscr{M} \equiv$ "the smallest σ-field that contains $\mathscr{F}_1 \times \mathscr{F}_2$, $\mathscr{F}'_1$ and $\mathscr{F}'_2$." Put $P_{(F_1,F_2,\psi)} =$ "the projection on a smallest closed subspace that contains $\{F_1(\Xi_1)F_2(\Xi_2)\psi \mid \Xi_1 \in \mathscr{F}_1, \Xi_2 \in \mathscr{F}_2\}$." By the same arguments in Lemma 1, we can see, from (8), that

$P_{(F_1,F_2,\psi)}F_1(\Xi_1)F_2(\Xi_2)=F_2(\Xi_2)F_1(\Xi_1)P_{(F_1,F_2,\psi)}$ $(\forall\Xi_1\in\mathcal{F}_1, \forall\Xi_2\in\mathcal{F}_2)$. Then, we can uniquely define the projection-valued probability M_i $(i=1,2)$ on $\mathcal{M}$ satisfying that, for each $i=1,2$,

$$M_i(\Xi_1\times\Xi_2)=P_{(F_1,F_2,\psi)}F_1(\Xi_1)F_2(\Xi_2) \qquad (\forall\Xi_1\in\mathcal{F}_1, \forall\Xi_2\in\mathcal{F}_2) \tag{9}$$

$$M_i(\Xi_i')=(I-P_{(F_1,F_2,\psi)})F_i(\Xi_i') \qquad (\forall\Xi_i'\in\mathcal{F}_i') \tag{10}$$

$$M_1(\Xi_2')=M_2(\Xi_1')=0 \qquad (\forall\Xi_2'\in\mathcal{F}_2', \forall\Xi_1'\in\mathcal{F}_1') \tag{11}$$

Define $f_i(x_1,x_2)=x_i$ $[\forall(x_1,x_2)\in X_1\times X_2]$, $f_i(x_i')=x_i'$ $(\forall x_i'\in X_i')$, and

$$f_1(x_2')=x_0\ (\forall x_2'\in X_2'), \qquad f_2(x_1')=y_0\ (\forall x_1'\in X_1')$$

where x_0 and y_0 are fixed constants in X_1 and X_2, respectively. Take M such that $M\approx_\psi M_1$. Then, $((Z,\mathcal{M},M), f: Z\to X_1\times X_2)$ is clearly a simultaneous measurement of observables $(X_1,\mathcal{F}_1,F_1)$ and $(X_2,\mathcal{F}_2,F_2)$ with respect to a state ψ.

Let ψ be a state of a system S in a Hilbert space V. Let $\mathbf{M}\,[\equiv((Z,\mathcal{M},M), f: Z\to X\times Y)]$ be a simultaneous measurement of observables $(X,\mathcal{F},F)$ and $(Y,\mathcal{G},G)$ with respect to a state ψ. We see, by Axiom 0, that the probability that z $(\in Z)$, the measurement value of the observable $(Z,\mathcal{M},M)$ for a system S with a state ψ, belongs to a set $f^{-1}(\Xi\times\Gamma)$ [i.e., the probability that $f(z)\in\Xi\times\Gamma(\subset X\times Y)$] is given by $\langle\psi, M(f^{-1}(\Xi\times\Gamma))\psi\rangle_V$ $[=\langle\psi, F(\Xi)G(\Gamma)\psi\rangle_V]$. So, the conditional probability $\mu_\psi(f_2(z)\in\Gamma\,|\,f_1(z)=x)$ that $f_2(z)$, the second component $f(z)$, belongs to a set $\Gamma(\in\mathcal{G})$ when we know that $f_1(z)=x$ is given intuitively by

$$\mu_\psi(f_2(z)\in\Gamma\,|\,f_1(z)=x)=\lim_{\Xi\to\{x\}}\frac{\langle\psi, M(f^{-1}(\Xi\times\Gamma))\psi\rangle_V}{\langle\psi, M(f^{-1}(\Xi\times Y))\psi\rangle_V}$$

$$=\lim_{\Xi\to\{x\}}\frac{\langle\psi, F(\Xi)G(\Gamma)\psi\rangle_V}{\langle\psi, F(\Xi)\psi\rangle_V} \tag{12}$$

Notice that $\mu_\psi(f_2(z)\in\Gamma\,|\,f_1(z)=x)$ [or $\mu_\psi(x,\Gamma)$] is independent of the choice of $(Z,\mathcal{M},M)$ and f. From a mathematical viewpoint, we must prepare the following definition:

Definition 1. Let ψ be a state in a Hilbert space V. Let $(X,\mathcal{F},F)$ and $(Y,\mathcal{G},G)$ be commutative observables with respect to ψ in V. Define a set $CP(\psi; (X,\mathcal{F},F), (Y,\mathcal{G},G))$ of all conditional probability $\mu_\psi[\mu_\psi(x,\Gamma)$, or more precisely $\mu_\psi(x,\Gamma\colon(X,\mathcal{F},F),(Y,\mathcal{G},G))]$ that satisfy the following conditions:

9. For each Γ $(\in\mathcal{G})$, $\mu_\psi(x,\Gamma)$ is $\mathcal{F}$-measurable as a function of x and $0\le\mu_\psi(x,\Gamma)\le 1$,
10. For each x $(\in X)$, $\mu_\psi(x,\cdot)$ is a probability measure on $(Y,\mathcal{G})$.

11. For each $\mathscr{F}$-measurable function $f: X \to \mathbf{R}$ and $\Gamma \in \mathscr{G}$,

$$\int_X f(x)\langle\psi, F(dx)G(\Gamma)\psi\rangle_V = \int_X f(x)\mu_\psi(x,\Gamma)\langle\psi, F(dx)\psi\rangle_V \qquad (13)$$

Notice that the existence of μ_ψ [i.e., $CP(\psi; (X, \mathscr{F}, F), (Y, \mathscr{G}, G)) \neq \varnothing$] is a well-known fact in probability theory under some conditions (for example, Y is a complete separable metric space and $\mathscr{G}$ is its Borel field (see, for example, Ash, 1972)). Also, the uniqueness in the following sense is assured:

12. If $\mu_1, \mu_2 \in CP(\psi; (X, \mathscr{F}, F), (Y, \mathscr{G}, G))$, then there exists a null set $N \in \mathscr{F}$ [i.e., $\langle\psi, F(N)\psi\rangle_V = 0$] such that $\mu_1(x,\Gamma) = \mu_2(x,\Gamma)$ for all $x \in X - N$ and $\Gamma \in \mathscr{G}$.

This is easily shown in what follows. Let $\{\Gamma_i^0 | \Gamma_i^0 \subset Y, i = 1, 2, \ldots\}$ be as defined in condition 8. Substituting $\Gamma = \Gamma_i$ in (13), we see that, for each i, there exists a null set N_i [i.e., $\langle\psi, F(N_i)\psi\rangle_V = 0$] such that $\mu_1(x,\Gamma_i) = \mu_2(x,\Gamma_i)$ for all $x \in X - N_i$. This implies that $\mu_1(x,\Gamma_i) = \mu_2(x,\Gamma_i)$ for all $i = 1, 2, \ldots$, and all $x \in X - \bigcup_{i=1}^{\infty} N_i$. Also, clearly, $\langle\psi, F(\bigcup_{i=1}^{\infty} N_i)\psi\rangle_V = 0$. Therefore, condition 12 holds.

Of course, these above arguments are rather ordinary. However, we now assume the following Postulate 3, which requires that the conditional probability $\mu_\psi(x,\Gamma)$ is effective not only when the simultaneous measurement of $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ for a system S with a state ψ is taken, but also when the measurement of an observable $(X, \mathscr{F}, F)$ for a system S with a state ψ is taken.

Postulate 3. Let ψ be a state of a system S in a Hilbert space V. Let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be commutative observables with respect to ψ in V such that $CP(\psi; (X, \mathscr{F}, F), (Y, \mathscr{G}, G)) \neq \varnothing$. Then, there exists $\hat{\mu}_\psi \in CP(\psi; (X, \mathscr{F}, F), (Y, \mathscr{G}, G))$ satisfying that, if we know that the "true" value of the observable $(X, \mathscr{F}, F)$ for this system is x_0 $(\forall x_0 \in X)$, then we can believe that the probability that y_0 $(\in Y)$, the "true" value of an observable $(Y, \mathscr{G}, G)$ for this system, belongs to a set Γ $(\in \mathscr{G})$ is given by $\hat{\mu}_\psi(x_0, \Gamma)$

We believe that there does not exist any experiment that is incompatible with Postulate 3, if Postulate 2 holds. In order to examine Postulate 3, we must use the simultaneous measurement of $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ with respect to ψ. However, any simultaneous measurement always assures (12). So, we can not find any experiment incompatible with Postulate 3.

Remark 3. Note that Postulates 1 and 2 are special cases of Postulate 3. Let $(X, \mathscr{F}, F)$, $(Y, \mathscr{G}, G)$, and f be as defined in Postulate 1. It is clear that $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ commute, since it holds that $F(\Xi)G(\Gamma) = F(\Xi)F(f^{-1}(\Gamma)) = F(f^{-1}(\Gamma))F(\Xi) = G(\Gamma)F(\Xi)$. Define $\hat{\mu}_\psi$ by

$$\hat{\mu}_\psi(x,\Gamma) = 1[f(x) \in \Gamma], \; = 0[f(x) \notin \Gamma]$$

Then, it clearly holds that $\hat{\mu}_\psi \in CP(\psi; (X, \mathcal{F}, F), (Y, \mathcal{G}, G))$. Also, we can similarly see that Postulate 2 is a special case of Postulate 3. Therefore, Postulates 1 and 2 should be understood in the sense of Postulate 3.

Now we shall attempt to extend Postulate 3 to the general case, that is, the case without the assumption that $(X, \mathcal{F}, F)$ and $(Y, \mathcal{G}, G)$ commute with respect to ψ.

Let ψ be a state of a system S in a Hilbert space V. Let $(X, \mathcal{F}, F)$ and $(Y, \mathcal{G}, G)$ be observables in V. Put

$$\mathcal{F}^\psi_{(Y,\mathcal{G},G)} = \{\Xi \in \mathcal{F} \mid G(\Gamma)F(\Xi)P_{(F,\psi)} = F(\Xi)G(\Gamma)P_{(F,\psi)}(\forall\Gamma \in \mathcal{G})\}$$

It is clear that $\mathcal{F}^\psi_{(Y,\mathcal{G},G)}$ is a σ-subfield of $\mathcal{F}$ and $(X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F)$ and $(Y, \mathcal{G}, G)$ commute with respect to ψ. Also, it is clear that $\varnothing \in \mathcal{F}^\psi_{(Y,\mathcal{G},G)}$ and $X \in \mathcal{F}^\psi_{(Y,\mathcal{G},G)}$, then $\mathcal{F}^\psi_{(Y,\mathcal{G},G)} \neq \varnothing$.

Suppose that $CP(\psi; (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G)) \neq \varnothing$. Note that this assumption is very weak (Ash, 1972). Assume that $x_0\ (\in X)$ is the "true" value of the observable $(X, \mathcal{F}, F)$ for a system S with a state ψ. So, since a map f: $(X, \mathcal{F}) \to (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)})$ such that $f(x) = x\ (\forall x \in X)$ is measurable, we can, by Postulate 1, believe that the "true" value of an observable $(X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F)$ for this system is the same x. Also, note that $(X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F)$ and $(Y, \mathcal{G}, G)$ commute with respect to ψ. So, applying Postulate 3, we can know that the probability $\mu_\psi(x, \Gamma: (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G))$ that $y\ (\in Y)$, the "true" value of an observable $(Y, \mathcal{G}, G)$ for this system, belongs to a set $\Gamma\ (\in \mathcal{G})$ when we know $x\ (\in X)$, the "true" value of the observable $(X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F)$ for this system. Therefore, we have the following postulate.

Postulate 4. Let ψ be a state of a system S in a Hilbert space V. Let $(X, \mathcal{F}, F)$ and $(Y, \mathcal{G}, G)$ be observables in V such that

$$CP(\psi; (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F), (\mathrm{Y}, \mathcal{G}, \mathrm{G})) \neq \varnothing$$

Then, there exists $\hat{\mu}_\psi \in CP(\psi; (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G))$ satisfying that, if we know that the "true" value of the observable $(X, \mathcal{F}, F)$ for this system S is $x_0(\forall_0 \in X)$, then we can believe that the probability that $y_0\ (\in Y)$, the "true" value of an observable $(Y, \mathcal{G}, G)$ for this system S, belongs to a set $\Gamma\ (\in \mathcal{G})$ is given by $\hat{\mu}_\psi(x_0, \Gamma)$.

Now we have the following main axiom.

Axiom 2 (Generalization of Axiom 1). Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathcal{F}, F)$ and $(Y, \mathcal{G}, G)$ be observables in V. Let μ_ψ be any element in $CP(\psi; (X, \mathcal{F}^\psi_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G))$. If we get $x_0\ (\in X)$ by the measurement of the observable $(X, \mathcal{F}, F)$ for this system S, then we can believe "almost surely" that the probability that $y_0\ (\in Y)$, the "true" value of the observable $(Y, \mathscr{G}, G)$ for this system S, belongs to a set $\Gamma\ (\in \mathscr{G})$ is given by $\mu_\psi(x_0, \Gamma)$

Remark 4. We must comment on the term "almost surely" in this axiom. Of course, if $\mu_\psi = \hat{\mu}_\psi$ (where $\hat{\mu}_\psi$ is defined in Postulate 4), we can say, by Postulate 4, that "we can believe (surely) that ..." in Axiom 2. On the contrary, if $\mu_\psi(x_0, \Gamma) \neq \hat{\mu}_\psi(x_0, \Gamma)$, then the judgement in the above axiom is not true. However, it holds, from (24), that there exists a null set $N \in \mathscr{F}^\psi_{(Y,\mathscr{G},G)}$ [i.e., $\langle \psi, F(N)\psi\rangle_V = 0$] such that $\hat{\mu}_\psi(x, \Gamma) = \mu_\psi(x, \Gamma)$ for all $x \in X - N$ and $\Gamma \in \mathscr{G}$. Also, it is clear, from Axiom 0, that the probability that $x_0\ (\in X)$, the value obtained by the measurement of the observable $(X, \mathscr{F}, F)$, belongs to N is 0 [i.e., $\langle \psi, F(N)\psi\rangle_V = 0$]. Therefore we can conclude that the judgement in Postulate 4 is true "almost surely."

Now we shall show that the "errors" of an approximate simultaneous measurement

$$\mathbf{M}\,[=(K, v, (X, \mathscr{F}, F), f = (f_0, f_1, \ldots, f_{N-1}))]$$

for $A_0, \ldots, A_{N-1}$ on a state u in a Hilbert space H can be naturally defined under our interpretation. Before this, we must prepare the following postulate.

Postulate 5. Let H and K be Hilbert spaces, and let V be a tensor Hilbert space of H and K, that is, $V = H \otimes K$. Let $(Y, \mathscr{G}, G)$ be an observable in H. Note that $(Y, \mathscr{G}, G \otimes I)$ is an observable in V where $(G\otimes I)(\Gamma)\otimes I(\forall \Gamma \in \mathscr{G})$ and I is an identity map on K. Let $u \in H$, $v \in K$ $(\|u\|_H = \|v\|_K = 1)$ and put $\psi = u \otimes v$. Then:

(i) If we know that the "true" value of the observable $(Y, \mathscr{G}, G)$ with respect to u is $y_0\ (\in Y)$, then we can believe that the "true" value of the observable $(Y, \mathscr{G}, G\otimes I)$ with respect to $u \otimes v$ is same $y_0\ (\in Y)$.

(ii) If we know that the "true" value $y_0\ (\in Y)$ of the observable $(Y, \mathscr{G}, G \otimes I)$ with respect to $u \otimes v$ is $y_0\ (\in Y)$, then we can believe that the "true" value of the observable $(Y, \mathscr{G}, G)$ with respect to u is same $y\ (\in Y)$.

Let H be a Hilbert space. Let $A_0, A_1, \ldots, A_{N-1}$ be self-adjoint operators (i.e., physical quantities) on a Hilbert space H. Let u be a state in H. Note that an approximate simultaneous measurement $\mathbf{M} = (K, v, (X, \mathscr{F}, F), f = (f_0, f_1, \ldots, f_{N-1}))$ for $A_0, A_1, \ldots, A_{N-1}$ with respect to u is equivalent to a simultaneous measurement $((X, \mathscr{F}, F), f = (f_0, f_1, \ldots, f_{N-1}))$ for $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1}$ with respect to $u \otimes v$ in $H \otimes K$, where $\hat{A}_k = \int_X f_k(x) F(dx)$. In this measurement $\mathbf{M}$, when we get $x_0\ (\in X)$ by the measurement of the observable $(X, \mathscr{F}, F)$ with respect to $u \otimes v$, we temporarily regard $f_k(x_0)$ as the "true" value of $A_k \otimes I$ with respect to $u \otimes v$

(so, by Postulate 5, the "true" value of A_k with respect to u). Also, when we get x_0 ($\in X$) by the measurement of the observable $(X, \mathscr{F}, F)$ with respect to $u\otimes v$, we can believe "almost surely" that the probability that y_0 ($\in \mathbf{R}$), the "true" value of an observable $A_k\otimes I = (\mathbf{R}, \mathscr{B}, E_{A_k\otimes I})$ for this system S, belongs to a set Γ ($\in \mathscr{B}$) is given by $\mu_{u\otimes v}(x_0, \Gamma\colon (X, \mathscr{F}^{u\otimes v}_{A_k\otimes I}, F), A_k\otimes I)$.

Therefore, it is natural to define $\delta_{\mathbf{M}}(A_k, u; x_0)$, the kth error when we get x_0 by the measurement of the observable $(X, \mathscr{F}, F)$ with respect to $u\otimes v$, by $[\int_{\mathbf{R}} |f_k(x_0)-\xi|^2 \mu_{u\otimes v}(x_0, d\xi)]^{1/2}$. Also, since the probability that $x_0 \in \Xi$ is given by $\langle u\otimes v, F(\Xi)(u\otimes v)\rangle$, it is reasonable to define $\bar{\delta}_{\mathbf{M}}(A_k, u)$, the kth average error in the measurement $\mathbf{M}$ with respect to a state u ($\in H$), by $[\int_X |\delta_{\mathbf{M}}(A_k, u; x_0)|^2 \langle u\otimes v, F(dx_0)(u\otimes v)\rangle]^{1/2}$.

Therefore, we get the following definition.

Definition 2. Let $A_0, A_1, \ldots, A_{N-1}$ be any physical quantities (i.e., self-adjoint operators) in a Hilbert space H. Let

$$\mathbf{M} = (K, v, (X, \mathscr{F}, F), f = (f_0, f_1, \ldots, f_{N-1}))$$

be an approximate simultaneous measurement of $A_0, A_1, \ldots, A_{N-1}$ in H. Then, $\delta_{\mathbf{M}}(A_k, u; x)$, the kth error when we get x by this measurement $\mathbf{M}$ with respect to a state u ($\in H$), is defined by $[\int_{\mathbf{R}} |f_k(x)-\xi|^2 \mu_{u\otimes v}(x, d\xi)]^{1/2}$ where

$$\mu_{u\otimes v}(x, \Gamma) \in CP(u\otimes v; (X, \mathscr{F}^{u\otimes v}_{A_k\otimes I}, F), A_k\otimes I)$$

Also, $\bar{\delta}_{\mathbf{M}}(A_k, u)$, the kth average error in the measurement $\mathbf{M}$ with respect to a state u ($\in H$), is defined by

$$[\int_X |\delta_{\mathbf{M}}(A_k, u; x)|^2 \langle u\otimes v, F(dx)(u\otimes v)\rangle]^{1/2}$$

Also, $\{\bar{\delta}_{\mathbf{M}}(A_k, u), |k = 0, 1, \ldots, N-1\}$ is called an average error in the measurement $\mathbf{M}$ with respect to a state u.

Now we have the following proposition.

Proposition 1. Let $A_0, A_1, \ldots, A_{N-1}$ be any physical quantities (i.e., self-adjoint operators) in a Hilbert space H. Let

$$\mathbf{M} = (K, v, (X, \mathscr{F}, F), f = (f_0, f_1, \ldots, f_{N-1}))$$

be an approximate simultaneous measurement of $A_0, A_1, \ldots, A_{N-1}$ in H. Assume that $\hat{A}_k$ $[=\int_X f_k(x) F(dx)]$ and $A_k\otimes I$ commute for each k. Then, the equalities $\Delta_{\mathbf{M}}(A_k, u) = \bar{\delta}_{\mathbf{M}}(A_k, u)$ $(k = 0, 1, \ldots, N-1)$ hold.

Proof. Fix k. Put $A_k\otimes I = (\mathbf{R}, \mathscr{B}, E_{A_k\otimes I})$. And put

$$\mathscr{F}^{u\otimes v}_{A_k\otimes I} = \{\Xi \in \mathscr{F} \mid E_{A_k\otimes I}(B) F(\Xi) P_{(F, u\otimes v)} = F(\Xi) E_{A_k\otimes I}(B) P_{(F, u\otimes v)} (\forall B \in \mathscr{B})\}$$

By the commutativity of $\hat{A}_k$ and $A_k \otimes I$, $f_k: X \to \mathbf{R}$ clearly is an $\mathcal{F}^{u\otimes v}_{A_k \otimes I}$-measurable function. Hence, we see, by (23), that

$$
\begin{aligned}
&|\bar{\delta}_{\mathbf{M}}(A_k, u)|^2 \\
&= \int_X (\delta_{\mathbf{M}}(A_k, u; x))^2 \langle u\otimes v, F(dx)(u\otimes v)\rangle \\
&= \int_{\mathbf{R}} \int_X |f_k(x) - \xi|^2 \mu_{u\otimes v}(x, d\xi) \langle u\otimes v, F(dx)(u\otimes v)\rangle \\
&= \int_{\mathbf{R}} \int_{(X, \mathcal{F}^{u\otimes v}_{A_k\otimes I})} |f_k(x) - \xi|^2 \mu_{u\otimes v}(x, d\xi) \langle u\otimes v, F(dx)(u\otimes v)\rangle \\
&= \int_{\mathbf{R}} \int_{(X, \mathcal{F}^{u\otimes v}_{A_k\otimes I})} |f_k(x) - \xi|^2 d\langle u\otimes v, F(dx) E_{A_k\otimes I}(d\xi)(u\otimes v)\rangle \\
&= |\Delta_{\mathbf{M}}(A_k, u)|^2 \quad \blacksquare
\end{aligned}
$$

Here we obtain the following theorem as a corollary of Theorem 1 and Proposition 1.

Theorem 2. Let A_0 and A_1 be a pair of conjugate observables in a Hilbert space H (i.e., symbolically, $A_0A_1 - A_1A_0 = i\hbar$). Let

$$\mathbf{M} = (K, v, (X, \mathcal{F}, F), f = (f_0, f_1))$$

be an approximate simultaneous measurement of A_0 and A_1 satisfying the additional condition 5. Assume that $\hat{A}_k$ [$=\int_X f_k(x) F(dx)$] and $A_k \otimes I$ commute for each k ($k = 0, 1$). Then, the following inequality holds:

$$\bar{\delta}_{\mathbf{M}}(A_0, u) \cdot \bar{\delta}_{\mathbf{M}}(A_1, u) \geq \hbar/2 \tag{14}$$

for all $u \in H$ ($\|u\|_H = 1$), where the left-hand side of this inequality is defined as $=\infty$ if $\Delta_{\mathbf{M}}(A_k, u) = \infty$ for some k.

If we take a standpoint within the interpretation proposed in this section, it is natural to consider that inequality (14) is just Heisenberg's uncertainty relation. However, if we study it within the Copenhagen interpretation, Theorem 2 does not seem to be suitable. Because there is no reason within the Copenhagen interpretation to define the error $\{\bar{\delta}_{\mathbf{M}}(A_k, u) \mid k = 0, 1\}$ as mentioned in Definition 2.

Remark 5. We are afraid that there may be various opinions about Postulate 2. For example, some physicists may assert that Postulate 2 holds only when $P_{(F,\psi)} = I$. However, it should be noted that, even if we agree to this assertion, our arguments (and results) in this note are essentially effective. Also, other physicists may assert that Postulate 2 holds only when

$P_{(F,\psi)} = F(X - \cup\{\Xi_i^0 \colon F(\Xi_i^0)\psi = 0\})$, where $\{\Xi_i^0\}_{i=1}^{\infty}$ is defined by condition 8. Of course, even in this case, our arguments (and results) are essentially effective. Our standpoint is that Postulate 2 should be assumed in quantum mechanics if there does not exist any experiment incompatible with it.

3. APPLICATION TO ANALYSIS OF A TRAJECTORY OF A PARTICLE

The interpretation proposed in the previous section seems to be very convenient if we want to discuss quantum mechanical theory. As a typical example, we shall analyze the trajectory of a particle (like "Wilson chamber") under this interpretation. It will be done by developing ideas of Arthurs and Kelly (1965) and She and Heffner (1966).

We shall consider a particle S in the one-dimensional real line $\mathbf{R}$, whose state function $u(t,\cdot)\,[\in H \equiv L^2(\mathbf{R}), -\infty < t < \infty]$ satisfies the following Schrödinger equation with a Hamiltonian $\mathcal{H} = -(\hbar^2/2m)\,\partial^2/\partial x^2$:

$$i\hbar \frac{\partial u(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 u(t,x)}{\partial x^2} = (\mathcal{H}u)(t) \qquad (-\infty < t < \infty) \tag{15}$$

$$u(0,x) = u(x)$$

Put $\theta > 0$ and $N \geq 2$ (integer). Let A be a position observable in H, that is, $(Au)(x) = xu(x)$.

Now we consider the approximate "simultaneous" measurement $\mathbf{M}$ of the positions of a particle S at time $t_k = \theta k$ $(k = 0, 1, 2, \ldots, N-1)$. Note that (15) is equivalent to the following Heisenberg kinetic equation of the time evolution of the observable A_t $(-\infty < t < \infty)$ in a Hilbert space H with a Hamiltonian H

$$-i\hbar \frac{dA_t}{dt} = \mathcal{H}A_t - A_t\mathcal{H} \qquad (-\infty < t < \infty) \tag{16}$$

Therefore we can consider that the measurement $\mathbf{M}$ is equivalent to the approximate simultaneous measurement of self-adjoint operators $\{A_{\theta k}\}_{k=0}^{N-1}$ for a particle S with a state $u(x) = u(0,x)$. An easy calculation shows that

$$A_t = U_{-t}AU_t = U_{-t}xU_t = x + \frac{\hbar t}{im}\frac{d}{dx} \tag{17}$$

where the one-parameter unitary group U_t $(= e^{-i\hbar^{-1}\mathcal{H}t})$ is represented by

$$(U_t u)(x) = u(t,x) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] u(\xi)\, d\xi \tag{18}$$

Here we see easily that

$$A_tA_s - A_sA_t = \frac{\hbar}{im}(t-s) \qquad (t, s \in \mathbf{R}) \tag{19}$$

Let $V = H \otimes K = H \otimes (\otimes_{k=1}^{N-1} H) = \otimes_{k=0}^{N-1} H = L^2(\mathbf{R}^N)$ and $\hat{U}_t = \otimes_{k=0}^{N-1} U_t$, that is, for all $\psi \in L^2(\mathbf{R}^N)$,

$$\hat{U}_t\psi(x_0, x_1, \ldots, x_{N-1}) = \left(\frac{m}{2\pi i\hbar t}\right)^{N/2} \int_{\mathbf{R}^N} \exp\left(\frac{im}{2\hbar t}\sum_{k=0}^{N}|x_k - \xi_k|^2\right) \times \psi(\xi_0, \ldots, \xi_{N-1})\, d\xi_0, \ldots, d\xi_{N-1} \tag{20}$$

Let α_{kn} $(k, n = 0, 1, \ldots, N-1)$ be real numbers such that

$$\sum_{n=0}^{N-1} \alpha_{kn}\alpha_{ln} = 0 \ (k \neq l)$$

and $\alpha_{k0} = 1$ $(\forall k)$. Define self-adjoint operators $\hat{A}_{\theta k}$ $(k = 0, 1, \ldots, N-1)$ in $V\,[\equiv L^2(\mathbf{R}^N)]$ by

$$\hat{A}_{\theta k} = \sum_{n=0}^{N-1} \alpha_{kn}\left(x_n + \frac{\hbar\theta k}{im}\frac{\partial}{\partial x_n}\right) \tag{21}$$

It is clear that $\hat{A}_{\theta k}$ $(k = 0, 1, 2, \ldots, N-1)$ commute. Also, for each k $(k = 0, 1, 2, \ldots, N-1)$, $\hat{A}_{\theta k}$ and $A_{\theta k} \otimes I[\equiv x_0 + (\hbar\theta k/im)\,\partial/\partial x_0]$ commute. We see, by (17), that

$$\hat{A}_{\theta k} = \hat{U}_{-\theta k}\left(\sum_{n=0}^{N-1} \alpha_{kn}x_n\right)\hat{U}_{\theta k}, \qquad A_{\theta k} \otimes I = \hat{U}_{-\theta k}x_0\hat{U}_{\theta k} \tag{22}$$

Then, the spectral measure $\hat{E}_{\theta k}$ of $\hat{A}_{\theta k}$ [i.e., $\hat{A}_{\theta k} = \int_{\mathbf{R}} \lambda \hat{E}_{\theta k}(d\lambda)$] is represented by

$$\hat{E}_{\theta k}(\Xi) = \hat{U}_{-\theta k}\chi\left(\Xi; \sum_{n=0}^{N-1} \alpha_{kn}x_n\right)\hat{U}_{\theta k} \qquad (\forall \Xi \in \mathcal{B}) \tag{23}$$

where $\chi(\Xi; y) = 1$ $(y \in \Xi)$, $=0$ $(y \notin \Xi)$, i.e., a characteristic function of Ξ. From the commutativity of $\{\hat{E}_{\theta k}\}_{k=0}^{N-1}$ (i.e., $\{\hat{A}_{\theta k}\}_{k=0}^{N-1}$), we can define an observable $(X, \mathcal{F}, F) = (\mathbf{R}^N, \mathcal{B}^N, F)$ in V where

$$F(\Xi_0 \times \Xi_1 \times \cdots \times \Xi_{N-1}) = \prod_{k=0}^{N-1} \hat{E}_{\theta k}(\Xi_k) \tag{24}$$

Put $u(x_0) = u_0(x_0)$ and

$$v(x_1, \ldots, x_{N-1}) = v_1(x_1) \cdots v_{N-1}(x_{N-1}) \in L^2(\mathbf{R}^{N-1})(\equiv K) \ (\|v\|_K = 1)$$

such that

$$\int_{\mathbf{R}} x_k|v_k(x_k)|^2\, dx_k = \int_{\mathbf{R}} \bar{v}_k(x_k)\frac{dv(x_k)}{dx_k}\, dx_k = 0, \qquad (k = 1, 2, \ldots, N-1) \tag{25}$$

Put f_k: X $(\equiv \mathbf{R}^N) \to \mathbf{R}$ $(k = 0, 1, \ldots, N-1)$ such that $f_k(x_0, \ldots, x_{N-1}) = x_k$. Note that $\hat{A}_{\theta k} = \int_X f_k(x) F(dx)$.

Now we shall show that $\mathbf{M} = (K, v, (X, \mathcal{F}, F), f = (f_0, \ldots, f_{N-1}))$ defined above is an approximate simultaneous measurement of $\{A_{\theta k}\}_{k=0}^{N-1}$ in H. The condition 1 (in the Introduction) is clear. Also, we see, by (25), that, for any $u \in H$,

$$
\begin{aligned}
&\langle u \otimes v, \hat{A}_{\theta k}(u \otimes v)\rangle_{H\otimes K} \\
&= \langle u_0 v_1 \cdots v_{N-1}, \sum_{n=0}^{N-1} \alpha_{kn} \left(x_n + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_n} \right) u_0 v_1 \cdots v_{N-1} \rangle \\
&= \alpha_{k0} \left\langle u_0, \left(x_0 + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_0} \right) u_0 \right\rangle + \sum_{n=1}^{N-1} \alpha_{kn} \left\langle v_n, \left(x_n + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_n} \right) v_n \right\rangle \\
&= \left\langle u_0, \left(x_0 + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_0} \right) u_0 \right\rangle \\
&= \langle u, A_{\theta k} u \rangle_H
\end{aligned} \tag{26}
$$

which implies that $D(A_{\theta k}) = D_v(\hat{A}_{\theta k})$, so conditions 2 and 3 hold.

Note that the probability that the measurement value $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_{N-1})$ obtained by the measurement $\mathbf{M}$ belongs to a set $\Xi_0 \times \Xi_1 \times \cdots \times \Xi_{N-1}$ is given by

$$
\left\langle u \otimes v, \prod_{k=0}^{N-1} \hat{E}_{\theta k}(\Xi_k)(u \otimes v) \right\rangle_{H\otimes K} \tag{27}
$$

Of course, this measurement value $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_{N-1})$ represents just the discrete trajectory of a particle S, though it includes errors.

Using Proposition 1, we can calculate the average error $\{\bar{\delta}_{\mathbf{M}}(A_{\theta k}, u)\}_{k=0}^{N-1}$ in the approximate simultaneous measurement $\mathbf{M}$ as follows:

$$
\begin{aligned}
&\bar{\delta}_{\mathbf{M}}(A_{\theta k}, u) \\
&= \| [\hat{A}_{\theta k} - A_{\theta k} \otimes I](u \otimes v) \|_{H\otimes K} \\
&= \left\| \sum_{n=1}^{N-1} \alpha_{kn} \left(x_n + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_n} \right) u_0 v_1 \cdots v_{N-1} \right\| \\
&= \left[\int_{\mathbf{R}^N} \left[\sum_{n=1}^{N-1} \alpha_{kn} \left(x_n + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_n} \right) \right. \right. \\
&\qquad \left. \left. \times v_1(x_1) \cdots v_{N-1}(x_{N-1}) \right]^2 dx_1 \cdots dx_{N-1} \right]^{1/2} \\
&= \left[\sum_{n=1}^{N-1} |\alpha_{kn}|^2 \int_{\mathbf{R}} \left| \left(x_n + \frac{\hbar \theta k}{im} \frac{\partial}{\partial x_n} \right) v_n(x_n) \right|^2 dx_n \right]^{1/2}
\end{aligned} \tag{28}
$$

Of course, if necessary, it is possible to obtain the error $\{\delta_{\mathbf{M}}(A_{\theta k}, u; \bar{x})\}_{k=0}^{N-1}$ when we get the measurement value $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_{N-1})$. However, it seems to be complicated in general since $\mathcal{F}^{u\otimes v}_{A_{\theta k}\otimes I}$ cannot be decided easily. So, it is convenient to use $\mathcal{F}^{(k)} \equiv \{\mathbf{R}^k \times \Xi \times \mathbf{R}^{N-k-1} | \Xi \in \mathcal{B}\}$ as a substitute for $\mathcal{F}^{u\otimes v}_{A_{\theta k}\otimes I}$. Since $(X, \mathcal{F}^{(k)}, F)$ and $A_{\theta k} \otimes I$ commute, we see that

$$
\begin{aligned}
&|\delta'_{\mathbf{M}}(A_{\theta k}, u, \bar{x})|^2 \\
&\equiv \int_{\mathbf{R}} |f_k(\bar{x}) - \xi|^2 \mu_{u\otimes v}(x, d\xi; (X, \mathcal{F}^{(k)}, F), A_{\theta k} \otimes I) \\
&= \int_{\mathbf{R}} |\bar{x}_k - \xi|^2 \lim_{\Xi \to \{\bar{x}_k\}} \frac{\langle u \otimes v, F(\mathbf{R}^k \times \Xi \times \mathbf{R}^{N-k-1}) E_{A_k \otimes I}(d\xi)(u \otimes v)\rangle_{H\otimes K}}{\langle u \otimes v, F(\mathbf{R}^k \times \Xi \times \mathbf{R}^k \times \mathbf{R}^{N-k-1})(u \otimes v)\rangle_{H\otimes K}} \\
&= \lim_{\Xi \to \{\bar{x}_k\}} \frac{\langle u \otimes v, \hat{E}_{\theta k}(\Xi) \hat{U}_{-\theta k}(\bar{x}_k - x_0)^2 \hat{U}_{\theta k}(u \otimes v)\rangle_{H\otimes K}}{\langle u \otimes v, \hat{E}_{\theta k}(\Xi)(u \otimes v)\rangle_{H\otimes K}} \\
&= \lim_{\varepsilon \to 0+} \Bigg(\Bigg\{ \int_{|\bar{x}_k - \Sigma_{n=0}^{N-1} \alpha_{kn} x_n| < \varepsilon} |\bar{x}_k - x_0|^2 |[\hat{U}_{\theta k}(u \otimes v)] \\
&\qquad \times (x_0, \ldots, x_{N-1})|^2 \, dx_0 \cdots dx_{N-1} \Bigg\} \\
&\qquad \times \Bigg\{ \int_{|\bar{x}_k - \Sigma_{n=0}^{N-1} \alpha_{kn} x_n| < \varepsilon} |[\hat{U}_{\theta k}(u \otimes v)](x_0, \ldots, x_{N-1})|^2 \, dx_0 \cdots dx_{N-1} \Bigg\}^{-1} \Bigg)
\end{aligned} \tag{29}
$$

We think that it is sufficient to use $\delta'_{\mathbf{M}}(A_{\theta k}, u, \bar{x})$ [or more roughly $\bar{\delta}_{\mathbf{M}}(A_{\theta k}, u)$] instead of $\delta_{\mathbf{M}}(A_{\theta k}, u, \bar{x})$ in most cases [though we have no proof that $\delta'_{\mathbf{M}}(A_{\theta k}, u, \bar{x}) = \delta_{\mathbf{M}}(A_{\theta k}, u, \bar{x})$]. Note also that we can calculate (27)-(29) numerically. Therefore, what is important is that we can make a computer simulation about this discrete trajectory $(\bar{x}_0, \ldots, \bar{x}_{N-1})$ of a particle S. Of course, this measurement $\mathbf{M}$ includes errors. However, we can estimate it!

Also, if we choose $u(x) = 1/[2(b-a)]^{1/2}$ for $x \in (-b, -a) \cup (a, b)$, $=0$ otherwise, we can make a computer simulation for a discrete trajectory of the two-slit experiment.

4. CONCLUSIONS

In this note, we proposed an interpretation in nonrelativistic quantum theory. Under this interpretation, we clarified that the uncertainties (i.e., Δq and Δp) in Heisenberg's uncertainty relation can be characterized as (average) errors in an approximate simultaneous measurement. Also, this interpretation seems to be very convenient for a discussion of quantum mechanical theory. As an example, we analyzed the (discrete) trajectory of a particle, and proposed a computer simulation for this trajectory.

REFERENCES

Abu-Zeid, O. (1987). *Physics Letters A*, **125**, 162-164.
Ali, S. T., and Emch, G. G. (1974). *Journal of Mathematical Physics*, **15**(2), 176-182.
Ali, S. T., and Prugovečki, E. (1976). *Journal of Mathematical Physics*, **18**(2), 219-228.
Arthurs, E., and Kelly, J. L., Jr. (1965). *Bell System Technical Journal*, **44**, 725-729.
Ash, B. R. (1972). *Real Analysis and Probability*, Academic Press, New York.
Busch, P. (1985). *International Journal of Theoretical Physics*, **24**, 63-92.
Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, New York.
Einstein, A., Podolsky, B., and Rosen, N. (1935). *Physical Review*, **47**, 777-780.
Holevo, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam.
Ishikawa, S. (1991). Uncertainty relations in simultaneous measurements for arbitrary observables, *Reports on Mathematical Physics*, to appear.
Prugovečki, E. (1981). *Quantum Mechanics in Hilbert Space*, Academic Press, New York.
Selleri, F. (1990). *Quantum Paradoxes and Physical Reality*, Kluwer Academic Publishers.
She, C. Y., and Heffner, H. (1966). *Physical Review*, **152**, 1103-1110.
Von Neumann, J. (1932). *Die Mathematische Grundlagen Der Quantenmechanik*, Springer-Verlag, Berlin.